Analysis 1

6 December 2023

Warm-up: see next slide

Find f' or df/dx.

Differentiate the functions whose letters are the start of your first or last name.

A) $x^2 - 5x + 27$

B) $\frac{1}{2} - x$

C) cx^3

Ć) $8\sin(x)$

D) $\dfrac{6x-7}{x^3}$

E) e^x

F) $x^2 - \cos(x)$

G) $6x^{-2}$

H) 1238

I) $\sqrt[3]{x}$

J) $x\cos(x)$

K) $5 - x^3$

L) $(x^2+1)(x^{10}-3)$

M) $\dfrac{2}{\sqrt{x}}$

N) $\dfrac{-2}{x^5}$

O) $x^{-1/9}$

P) $\sqrt{\sqrt{x}}$

Q) $7x^2 + 5 + 3x^{-1}$

R) $x^4 - x^3 + x^2 - x + 1$

S) $5 + \sqrt{5}$

Ś) $7 + 2x$

Ș) $7 \cdot 2x$

T) 7^{2x}

U) $3\sin(x) + 2\cos(x)$

V) $\cos(x) + \sqrt{x}$

W) $\cos(x) \cdot \sqrt{x}$

X) $\cos(x) \cdot \sin(x)$

Y) $\sqrt{x^5}$

Z) $6x^{-2} + 5x^2$

Ż) $100x^{99}$

Differentiate the functions whose letters are the start of your first or last name.

(do it again!)

A) $2x - 5$

B) -1

C) $3c\,x^2$

Ć) $8\cos(x)$

D) $\dfrac{21 - 12x}{x^4}$

E) e^x

F) $2x + \sin(x)$

G) $-12x^{-3}$

H) 0

I) $\frac{1}{3}x^{-2/3}$

J) $\cos(x) - x\sin(x)$

K) $-3x^2$

L) $(x^2 + 1)10x^9 + 2x(x^{10} - 3)$
$= 12x^{11} + 10x^9 - 6x$

M) $\dfrac{-1}{x^{3/2}}$

N) $\dfrac{10}{x^6}$

O) $\frac{-1}{9}x^{-10/9}$

P) $\frac{1}{4}x^{-3/4}$

Q) $14x - 3x^{-2}$

R) $4x^3 - 3x^2 + 2x - 1$

S) 0

Ś) 2

Ș) 14

T) $2\ln(7)\cdot 7^{2x}$

U) $3\cos(x) - 2\sin(x)$

V) $-\sin(x) + \frac{1}{2}x^{-1/2}$

W) $\frac{1}{2}x^{-1/2}\cos(x) - \sin(x)\sqrt{x}$

X) $\cos(x)^2 - \sin(x)^2$

Y) $\frac{5}{2}x^{3/2}$

Z) $-12x^{-3} + 10x$

Ż) $100x^{99}$

Answers:

A) 2

B) 0

C) $6c\,x$

D) $-8\sin(x)$

Ć) $\dfrac{36x-84}{x^5}$

E) e^x

F) $2+\cos(x)$

G) $36x^{-4}$

H) 0

I) $\frac{-2}{9}x^{-5/3}$

J) $-2\sin(x) - x\cos(x)$

K) $-6x$

L) $132x^{10}+90x^8-6$

M) $\dfrac{3}{2x^{5/2}}$

N) $\dfrac{-60}{x^7}$

O) $\frac{10}{81}x^{-19/9}$

P) $\frac{-3}{16}x^{-7/4}$

Q) $14+6x^{-3}$

R) $12x^2-6x+2$

S) 0

Ś) 0

Ş) 0

T) $4(\ln 7)^2\cdot 7^{2x}$

U) $-3\sin(x)-2\cos(x)$

V) $-\cos(x)-\frac{1}{4}x^{-3/2}$

W) $-\frac{1}{4}x^{-3/2}\cos(x) - x^{-1/2}\sin(x)$
$-x^{1/2}\cos(x)$

X) $-4\sin(x)\cos(x)$

Y) $\frac{15}{4}x^{1/2}$

Z) $36x^{-4}+10$

Ż) $9900x^{98}$

Seeing f′ and f″ in graphs

Last time

<u>Monotonicity</u>

- If $f' > 0$ then f is "increasing",
- If $f' < 0$ then f is "decreasing".
- A point where f' *is zero* or *doesn't exist* is a "critical point".

<u>Concavity</u>

- If $f'' > 0$ then f is "concave up",
- If $f'' < 0$ then f is "concave down".
- A point where f'' *changes sign* is an "inflection point".

Min/max tasks

Last time

"Find the absolute extremes of $f(x)$ on the closed interval $[a, b]$."

HOW TO DO IT:

- Compare values of f at CPs and endpoints.

"Find the local extremes of $f(x)$."

"Find and classify the critical points of $f(x)$."

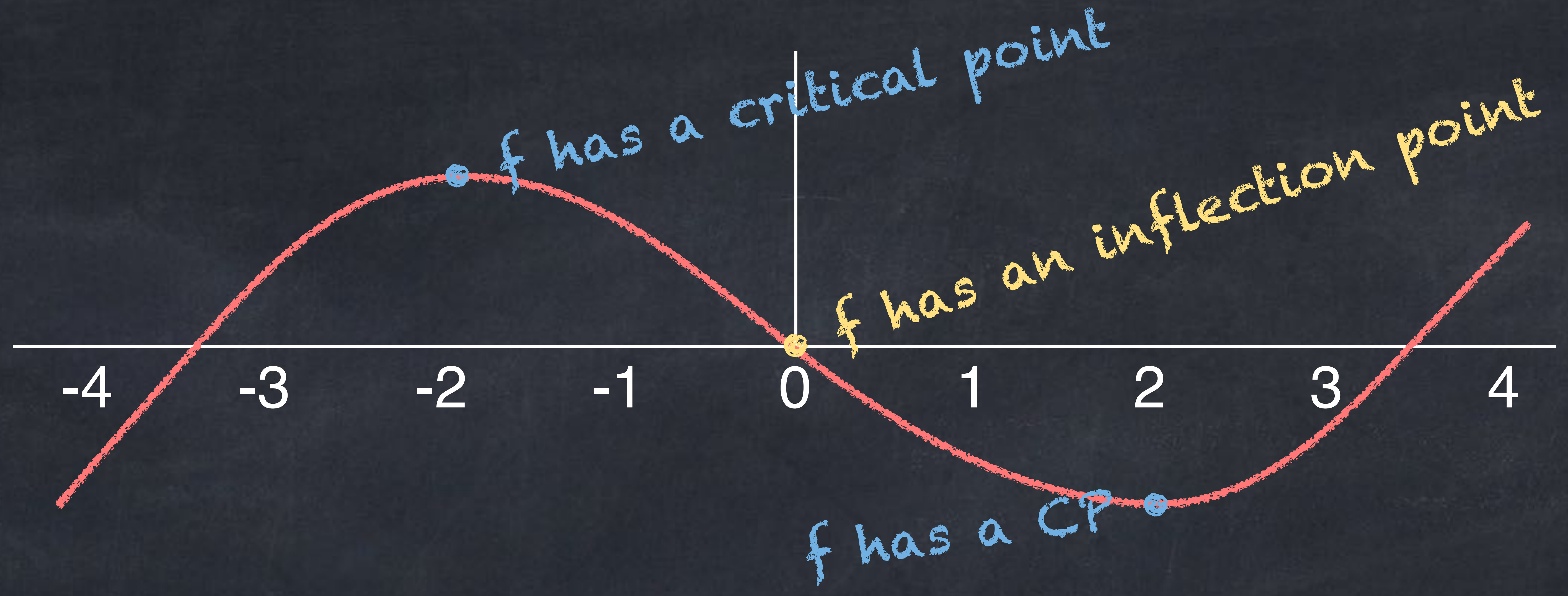
HOW TO DO IT:

- 1^{st} Deriv. Test: Use values of f' in between CPs, and far left, and far right.

  OR

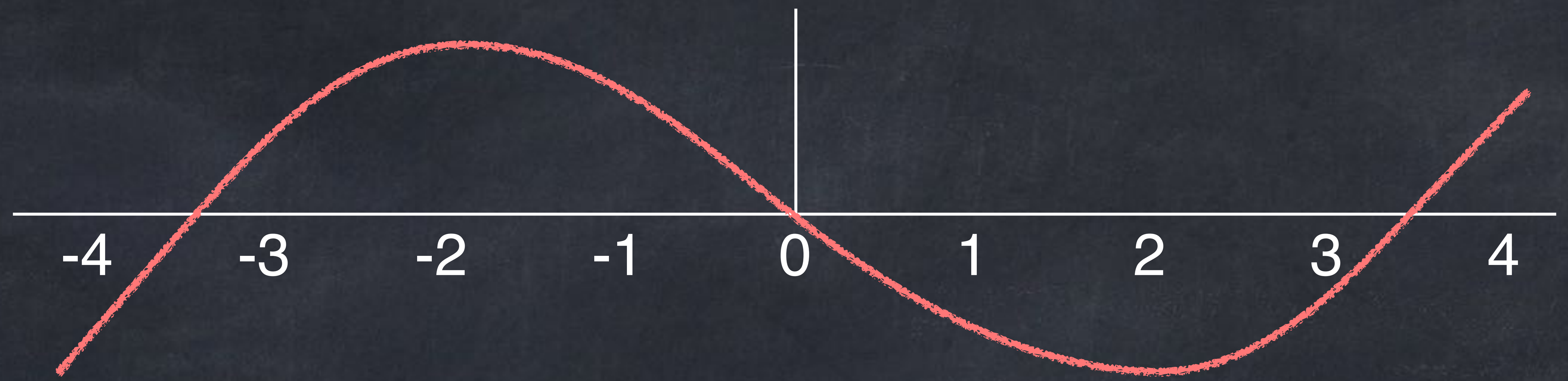
- 2^{nd} Deriv. Text: Use values of f'' at each CP.

For $f(x) = \frac{1}{3}x^3 - 4x$,



	f is negative	f is positive	f is negative	f is pos.
f′ tells us	f is increasing	f is decreasing	f is increasing	
f″ tells us	f is concave down	f is concave up		

For $f(x) = \frac{1}{3}x^3 - 4x$,



<table>
<tr><td></td><td>f is negative</td><td colspan="2">f is positive</td><td colspan="2">f is negative</td><td>f is pos.</td></tr>
<tr><td>f′ tells us</td><td colspan="2">f is increasing
f′ is positive</td><td colspan="2">f is decreasing
f′ is negative</td><td colspan="2">f is increasing
f′ is positive</td></tr>
<tr><td>f″ tells us</td><td colspan="3">f is concave down
f′ is decreasing
f″ is negative</td><td colspan="3">f is concave up
f′ is increasing
f″ is positive</td></tr>
</table>

Task: Classify the critical points of

$$f(x) = x^4 - \frac{64}{3}x^3 + 154x^2 - 392x$$

and list its inflection points.

Possible hints:

$$x^4 - \frac{64}{3}x^3 + 154x^2 - 392x = 0 \quad \rightarrow \quad x = 0,\ x = \frac{28}{3}$$

$$4x^3 - 64x^2 + 308x - 392 = 0 \quad \rightarrow \quad x = 2,\ x = 7$$

$$12x^2 - 128x + 308 = 0 \quad \rightarrow \quad x = \frac{11}{3},\ x = 7$$

$$24x - 128 = 0 \quad \rightarrow \quad x = \frac{16}{3}$$

Derivative Rules

We have several rules that can help us find derivatives of functions (without doing limits).

Individual functions:

- Deriv. of constant is zero
- Powers
- Sine and cosine
- Exponential
- Logarithm ← new today

Combining functions:

- Sum
- Product
- Quotient ← new today
- Composition ← new today

Summary of rules:

- $(c)' = 0$
- $(x^c)' = c\,x^{c-1}$
- $(c^x)' = c^x \ln(c)$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\ln x)' = ???$
- $(c f)' = c\,(f')$
- $(f + g)' = f' + g'$
- $(f g)' = fg' + f'g$

We do not need to use f and g as the names of the functions, and we do not need to use x as the variable.

- If $u = 10x^3 + 1$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = 30x^2$.
- If $u = t\cos(t)$ then $\frac{\mathrm{d}u}{\mathrm{d}t} = \cos(t) - t\sin(t)$.
- If $y = \sin(v)$ then $\frac{\mathrm{d}u}{\mathrm{d}v} = \cos(v)$.
- If $f = u^2$ then $\frac{\mathrm{d}f}{\mathrm{d}u} = 2u$.

We have seen how to do derivatives of $f(x) + g(x)$ and $f(x) - g(x)$ and $f(x) \cdot g(x)$. We will look at $\frac{f(x)}{g(x)}$ later.

There is one other important way to combine functions: the **composition** of $f(x)$ and $g(x)$ is the function $f(g(x))$, which can also be written as $f \circ g$.

Examples of compositions:

- $\sin(x^2)$
- $\sqrt{\cos(x)}$
- $\ln(x^3 + 8)$
- $e^{(-x^2)}$
- $(5 + \cos(x))^3$
- $\ln(\sin(e^x))$

The "Chain Rule" will let us find derivatives of all of these functions!

Before learning the general formula for $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right]$, let's look at a composition that we can already differentiate with other methods:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(10x^3+1)^2\right] = ?$$

We will answer this three different ways:

- By expanding $(10x^3+1)^2 = 100x^6 + 20x^3 + 1$.
- By the PRODUCT RULE because $(10x^3+1)^2 = (10x^3+1)\cdot(10x^3+1)$.
- By the CHAIN RULE (new)!

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(10x^3 + 1)^2\right] = \underbrace{2(10x^3+1)}_{2u \,=\, \frac{\mathrm{d}f}{\mathrm{d}u}}\underbrace{(30x^2)}_{\frac{\mathrm{d}u}{\mathrm{d}x}}$$

We do not need to use f and g as the names of the functions, and we do not need to use x as the variable.

- If $u = 10x^3 + 1$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = 30x^2$.
- If $u = t\cos(t)$ then $\frac{\mathrm{d}u}{\mathrm{d}t} = \cos(t) - t\sin(t)$.
- If $y = \sin(v)$ then $\frac{\mathrm{d}u}{\mathrm{d}v} = \cos(v)$.
- If $f = u^2$ then $\frac{\mathrm{d}f}{\mathrm{d}u} = 2u$.

Chain Rule

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Although $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not really a fraction, the idea of canceling out parts of fractions is a nice way to remember one of the official Chain Rule formulas.

Chain Rule

- $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left[f \circ g\right] = f'(g) \cdot g'$

You do *not* need to know *any* of these formulas.

You only need to be able to *use* the Chain Rule to find derivatives of functions.

Task 1: Differentiate $(\sin(x))^4$.

Chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If $f(x) = x^4$, then $f'(x) = 4x^3$

$$f'(3) = 4(3)^3 = 108$$

$$f'(w) = 4w^3$$

$$f'(\sin(x)) = 4\sin(x)^3$$

We will also need $g'(x) = \cos(x)$, the deriv. of $\sin(x)$.

According the formula in the box, the derivative of $f(g(x)) = \sin(x)^4$ is $4\sin(x)^3 \cdot \cos(x)$.

For the example $(\sin(x))^4$, we call $\sin(x)$ the "inside function" and we call $(\ )^4$ the "outside function".

Task 2: Find the derivative of $(4x^2 - 8x + 9)^{50}$.

Task 3: Find the derivative of $\sin(4x^2 - 8x + 9)$.

Task 4: Find the derivative of $x^3 e^x + \sin(x^2)$.

- Use the SUM RULE first.
- Then use the PRODUCT RULE for $\frac{d}{dx}\left[x^3 e^x\right]$.
- And use the CHAIN RULE for $\frac{d}{dx}\left[\sin(x^2)\right]$.

sin(x) is the "outside function"

x^2 is the "inside function"

(derivative of x^3e^x) + (deriv. of $\sin(x^2)$)

$$= (x^3)\cdot\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[x^3\right]\cdot(e^x) + \cos(x^2)\cdot 2x$$

derivative of inside function

derivative of outside function

inside function

$$= x^3 e^x + 3x^2 e^x + 2x\cos(x^2)$$

or $(x+3)x^2e^x + 2x\cos(x^2)$

Task 5a: Differentiate $(3x - 7)(2x + 1)^5$.

- Use the PRODUCT RULE first.
- Then use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^5\right]$.

$$(3x-7)(\underline{\text{deriv. of } (2x+1)^5}) + (3)(2x+1)^5$$

$$= (3x-7)(5(2x+1)^4 \cdot 2) + (3)(2x+1)^5$$

$$= (2x+1)^4 \left(2(3x-7) + 3(2x+1) \right)$$

$$= (2x+1)^4 (36x - 67)$$

Task 5b: Differentiate $(3x-7)(2x+1)^{-1}$.

- Use the PRODUCT RULE first.
- Then use the CHAIN RULE for $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^{-1}\right]$.

$$(3x-7)(-(2x+1)^{-2})(2) + (3)(2x+1)^{-1}$$

With some algebra, this is also $\boxed{\dfrac{17}{(2x+1)^2}}$.

This is one way to differentiate $\dfrac{3x-7}{2x+1}$. There is also "the quotient rule".

The Quotient Rule $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$ can be helpful, but you can always use Product and Chain instead, like we did for $\frac{3x-7}{2x+1} = (3x-7)(2x+1)^{-1}$.

Example: Find the derivative of $\tan(x)$.

- You should know that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Simplify $e^{\ln(x)}$.

- This is exactly x.

What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\ln(x)}\right]$?

- Since this is $\frac{\mathrm{d}}{\mathrm{d}x}\left[x\right]$, it must be 1.
- According to the Chain Rule, it's also $e^{\ln(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right]$, which is $x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right]$.
- That means $x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right] = 1$.
  - Dividing both sides of that equation by x gives $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\right] = \frac{1}{x}$.

Derivative formulas

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$1\,/\,x$

You should memorize these!

a^x	$a^x \ln(a)$
$\tan(x)$	$\sec(x)^2$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$

Maybe these too.

Derivative formulas

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$1 / x$

Constant Multiple: $(cf)' = cf'$

Sum Rule: $(f+g)' = f' + g'$

Product Rule:

$$(fg)' = fg' + f'g$$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Chain Rule:

$$\big(f(g)\big)' = f'(g) \cdot g'$$